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LETTER TO THE EDITOR

A note on a capillarity model and the nonlinear Schrödinger equation

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Abstract. A system of two equations governing the irrotational flow of a capillary fluid is shown, for a particular class of free energy functions, to reduce to a nonlinear Schrödinger equation.

1. Introduction

In continuum mechanics, the adoption of model constitutive laws for which the governing equilibrium or dynamic equations reduce to tractable form is well established. In gas dynamics, this approach goes back at least to the pioneering work by Chaplygin in 1904 on gas jets [1]. Subsequent extensive developments on the application of model laws in gas dynamics are catalogued by Dombrovskii [2]. A comprehensive treatment of the subject based on Bäcklund transformations is due to Loewner [3] and described, in detail, in the monograph by Rogers and Shadwick [4]. Deep connections between Loewner's work and soliton theory have been established by Konopelchenko and Rogers [5, 6] and Schief and Rogers [7].

Here, we note another link between fluid dynamics and soliton theory involving model laws, in this case for a capillarity system recently introduced by Antanovskii [8]. It is shown that, for a particular class of free energy functions, the dynamic equations for this capillarity system reduce to a nonlinear Schrödinger (NLS) equation. In $(1+1)$ dimensions, a reduction to the integrable cubic Schrödinger equation is obtained.

2. A class of model laws in capillarity and the NLS equation

Here, we consider the following system governing the flow of an inviscid, isothermal capillary fluid (Antanovskii [8]):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \tag{1}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = \boldsymbol{g} - \nabla \left[\frac{\delta(\rho f)}{\delta \rho} \right] \tag{2}$$

where if $\alpha = \frac{1}{2}|\nabla\rho|^2$ then

$$f = f(\rho, \alpha) \tag{3}$$

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denotes the specific free energy. In the above

$$\frac{\delta\Phi}{\delta\rho} \equiv \frac{\partial\Phi}{\partial\rho} - \nabla\cdot\left(\frac{\partial\Phi}{\partial\alpha}\nabla\rho\right) \tag{4}$$

designates the variational derivative. The quantity

$$\zeta = \frac{\delta[\rho f]}{\delta\rho} \tag{5}$$

is the chemical potential of the liquid–vapour system; ρ denotes the density of the fluid, $\boldsymbol{v}$ its velocity and t the time.

In the case of irrotational flow with conservative external force $\boldsymbol{g}$, there exist potentials ϕ and Π such that

$$\boldsymbol{v} = \nabla\phi \qquad \boldsymbol{g} = \nabla\Pi.$$

Hence the governing system (1) and (2) reduces to

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\nabla\phi) = 0 \tag{6}$$

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{\delta}{\delta\rho}(\rho f) - \Pi = \mathbb{B}(t). \tag{7}$$

It is noted that the arbitrary $\mathbb{B}(t)$ in the Bernoulli integral (7) may be absorbed into the potential ϕ, and, accordingly, is henceforth set zero.

On introduction of

$$q = \rho^{1/2}\,\mathrm{e}^{\mathrm{i}\phi/2} \tag{8}$$

it is seen that

$$\frac{\partial q}{\partial t} = \frac{1}{2}\left[\rho^{-1/2}\frac{\partial\rho}{\partial t} + \mathrm{i}\rho^{1/2}\frac{\partial\phi}{\partial t}\right]\mathrm{e}^{\mathrm{i}\phi/2} \tag{9}$$

$$\Delta q = \frac{1}{2}\left[\rho^{-1/2}\Delta\rho - \frac{1}{2}\rho^{-3/2}|\nabla\rho|^2 + \mathrm{i}\rho^{1/2}\Delta\phi + \mathrm{i}\rho^{-1/2}\nabla\rho\cdot\nabla\phi - \frac{1}{2}\rho^{1/2}|\nabla\phi|^2\right]\mathrm{e}^{\mathrm{i}\phi/2}. \tag{10}$$

Hence on use of the continuity equation (6) and Bernoulli integral (7) we obtain

$$\mathrm{i}\frac{\partial q}{\partial t} + \Delta q = \frac{1}{2}\left[\frac{\delta}{\delta\rho}(\rho f) + \frac{\Delta\rho}{\rho} - \frac{1}{2}\frac{|\nabla\rho|^2}{\rho^2} - \Pi\right]q \tag{11}$$

or, equivalently, if $\Pi = 0$, as

$$\mathrm{i}\,\frac{\partial q}{\partial t} + \Delta q = \frac{1}{2}\frac{\delta}{\delta\rho}\left[\rho f(\rho,\alpha) - \frac{\alpha}{\rho}\right]q. \tag{12}$$

If attention is restricted to free energy functions of the type

$$f = \frac{1}{2\rho^2}|\nabla\rho|^2 + H(\rho) \tag{13}$$

where $H(\rho)$ is arbitrary, then it is seen that (12) reduces to the nonlinear Schrödinger equation

$$\mathrm{i}\frac{\partial q}{\partial t} + \Delta q + J(|q|)q = 0 \tag{14}$$

where

$$J(|q|) = -\tfrac{1}{2}[\rho H(\rho)]' \qquad |q| = \sqrt{\rho}. \tag{15}$$

In this case, the capillary pressure adopts the form

$$p \equiv \rho^2 \frac{\delta f}{\delta \rho} = \rho^2 H'(\rho) - \rho \Delta(\log \rho). \tag{16}$$

In general, the NLS equation (14) is not integrable. However, in $(1+1)$ dimensions, if $J = \nu |q|^2$ then the integrable cubic Schrödinger equation

$$\mathrm{i}\frac{\partial q}{\partial t} + \frac{\partial^2 q}{\partial x^2} + \nu |q|^2 q = 0 \tag{17}$$

results. Accordingly, exact solutions of (17), such as the multi-kink soliton solutions generated by Boiti *et al* [9] via Bäcklund transformations are readily obtained for the $(1+1)$-dimensional capillarity system with model free energy functions

$$f = \frac{\alpha}{\rho^2} - \frac{\mu}{\rho} - \nu\rho \qquad \mu, \nu \in \mathbb{R}. \tag{18}$$

3. Conclusion

Hydrodynamic interpretation of Schrödinger-type equations has its origin in work as far back as 1926 by Madelung [10] and has subsequently been the subject of investigation by, amongst others, Degtyarev and Krylov [11] and Perrie [12]. Here, capillarity is encapsulated in the hydrodynamic system. It is important to stress that it is only this presence of capillarity that allows exact reduction to the NLS equation.

References

[1] Chaplygin S A 1904 *Sci. Mem. Moscow. Univ. Math. Phys.* **21** 1–121
[2] Dombrovskii G A 1964 *Approximation Methods in the Theory of Plane Adiabatic Gas Flow* (Moscow: Nauka)
[3] Loewner C 1950 *Nat. Advis. Coun. Aeronaut. Tech. Notes* **2065** 1–56
[4] Rogers C and Shadwick W F 1982 *Bäcklund Transformations and their Applications* (New York: Academic)
[5] Konopelchenko B and Rogers C 1991 *Phys. Lett.* **158A** 391–7
[6] Konopelchenko B and Rogers C 1993 *J. Math. Phys.* **34** 214–42
[7] Schief W K and Rogers C 1997 *Stud. Appl. Math.* to be published
[8] Antanovskii L K 1996 *Phys. Rev.* E **54** 6285–90
[9] Boiti M, Laddomada C and Pempinelli F 1981 *Nuovo Cimento* **65** 248–53
[10] Madelung E 1926 *Z. Phys.* **40** 322–6
[11] Degtyarev L M and Krylov V V 1978 USSR *Comput. Maths. Math. Phys.* **17** 172–9
[12] Perrie W A 1982 *Proc. R. Soc., London* A **379** 289–304